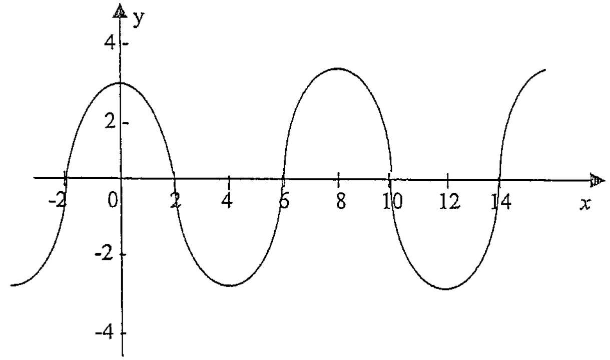
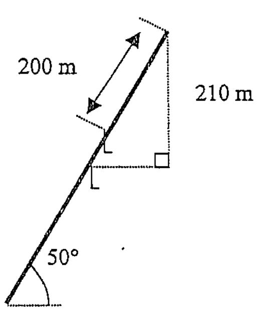
Trig - Miscellaneous Bursary Questions

1 (c) Write down the period and the amplitude of the function whose graph is shown below.



2(b) A chairlift rises in a straight line up a mountainside at an angle of 50^0, as shown in the diagram. When one chair has 200m to travel to the peak, a second is at a vertical distance of 210 m below the peak. Calculate the distance between the chairs.



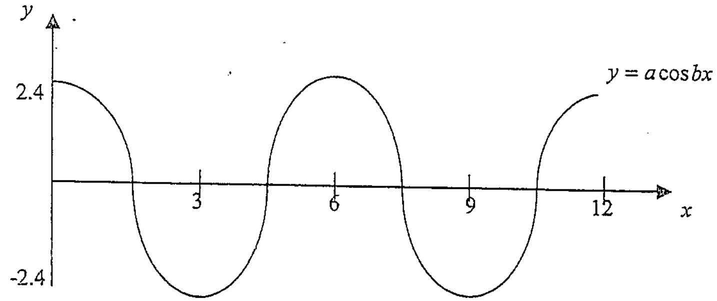
(2)

(c) Suppose $\tan\theta = t$. Prove that $\sin 2\theta = \dfrac{2t}{1+t^2}$. (2)

(d) A fishing boat runs aground one night, at a place where the water level, relative to its mean level, is given in metres by $h = 2.7\cos\dfrac{\pi t}{6}$ where t is in hours.

1. Sketch one full cycle of the graph of this function, labelling the axes clearly.
2. The boat can be refloated when the water level is 2.1 m above its mean level. Draw on the graph a line representing this water level and also show on the graph the first time after a low tide when refloating can occur.
3. If there is a low tide at 10.30 am the following morning, find to the nearest minute the time when refloating can occur. (4)

3 (a) Shown below is the graph of $y = a\cos bx$.



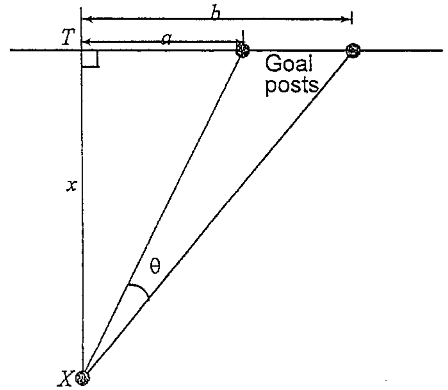
1. State the values of the amplitude and period of this function.
2. State the values of a and b.

(b) Write $2\cos 3D\sin D$ as a sum of two trigonometric functions.

(c) Prove that $(\tan\theta + \sec\theta)^2 = \dfrac{1+\sin\theta}{1-\sin\theta}$.

Question 25 *(Answer p195)*

A try in rugby is scored outside the posts at point T. The conversion attempt will be kicked from point X at a distance x back from the goal line. The player taking the kick would like to know the value of x which maximises the angle θ between the posts.

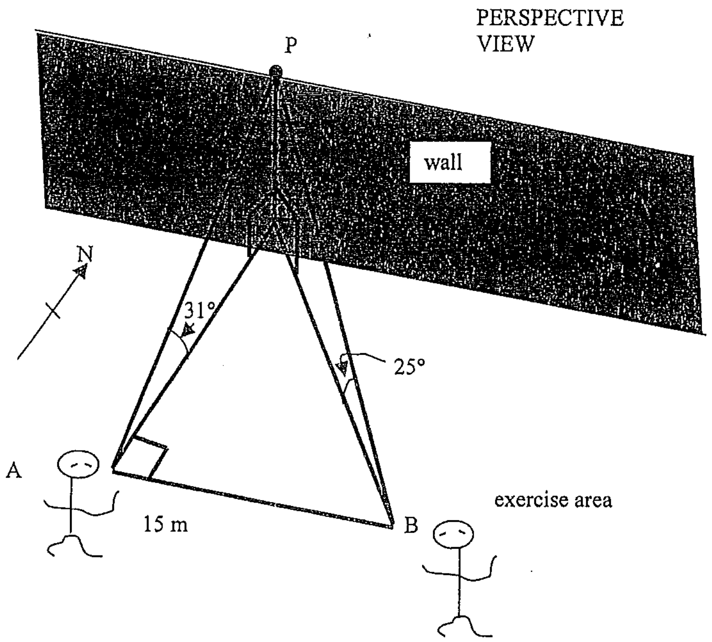


(a) Show that $\tan(\theta) = \dfrac{x(b-a)}{x^2+ab}$

where a and b are the distances from the try position to each post.

4 (c) Solve the equation $\sin 5x = \sin x$ for $0 \le x \le \pi$.

(d) Some desperate prisoners are planning an escape from their prison. They have secretly constructed a clinometer and have had a measuring tape smuggled in. They plan to scale the prison wall by using a rope made of bed sheets attached to a grappling hook. They need to calculate the height of the wall so that they can determine how many sheets they must tie together to make their rope the full height of the wall at the crossing point P. Each sheet contributes 2.1 m to the length of the rope. From a point A in the prison yard, due south of P, a prisoner measures the angle of elevation of P to be 31^0. From a point B, 15 m due east of A, he finds the angle of elevation of P is 25^0. The eye of this prisoner is 1.7 m above the ground.



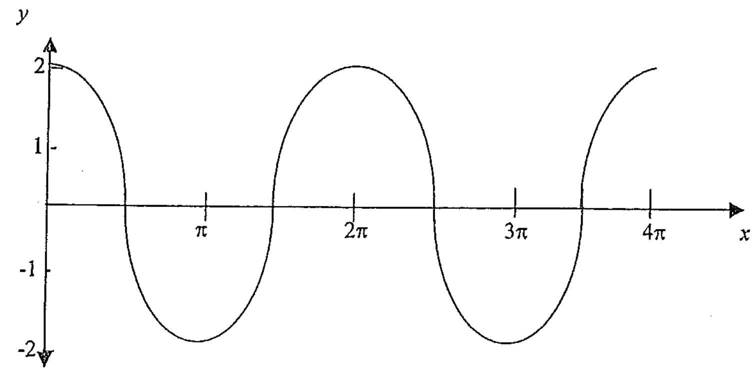
Find the vertical height of the wall to the nearest 0.1 m, and hence decide the number of sheets which are needed for the rope.

5 (a) 1. Solve $\cos x = -0.6$ where x is in radians and $0 \le x \le 2\pi$

2. Solve $\sin 2x = 0.5$ where $0^0 \le x \le 180^0$.

(b) 1. State the values of A, B and C, if the graph shown below shows the function:

$$y = A \sin B(x + C).$$



2. Prove that $\sec^2 x + \operatorname{cosec}^2 x = \sec^2 x \operatorname{cosec}^2 x$

7 (a) Solve for x. Give all answers in degrees.

1. $\operatorname{cosec} x = 2, \quad 0 \le x \le 180°$.

2. $\tan 2x = 2, \quad 0 \le x \le 180°$.

(b) Some stars have a brightness that periodically increases and decreases. The brightness, B, of one such star can be modelled by the equation:

$$B = 3.9 + 0.33 \sin\left(\frac{2\pi t}{5.6}\right)$$

where t is the time measured in days.

1. What is the maximum brightness of the star?

2. What is the minimum brightness of the star?

3. How many days elapse between successive times of maximum brightness?

(c) Prove that $\tan^2 x - \sin^2 x = \tan^2 x . \sin^2 x$.

10(b) Solve for x. Give all answers in degrees.

1. $\tan x = -0.5,\qquad -180^\circ \le x \le 180^\circ.$
2. $\sec(2x) = 2,\qquad 0^\circ \le x \le 180^\circ.$

(c) Prove that $\cot^2 x - \cos^2 x = \cot^2 x \cos^2 x$

11 (a) Given that $\tan A = \frac{3}{4}$ and $\tan B = \frac{3}{5}$, find the value of $\tan(A + B)$. (2)

(b) Find all the values of x that satisfy the equation

$$\cos^2 x - \sin x \cos x = 0, \qquad 0 \le x \le 2\pi$$ (3)

(c) Awarua Inlet is a tidal estuary in Abel Tasman National Park. The depth of water in the middle of the channel changes as the tide comes in and out. This depth can be modelled by a cosine function of the form $y = A \cos Bt + C$. Successive high tides occur every 12½ hours. The maximum depth of the water at high tide is 2.0 m while the minimum depth at low tide is 0.6 m. It is safe to walk across the inlet provided the depth of water is not more than 1 m. High tide on a particular day occurs at 10.00 am. Between what times in the afternoon of that day will it be safe to walk across the inlet? (4)

Question 13 *(Answer p188)*

Prove that $\frac{\sin(3x)}{\sin(x)} - \frac{\cos(3x)}{\cos(x)} = 2$ for all values of x where $\sin(x)\cos(x) \ne 0$.

Question 14 *(Answer p188)*

There is a flash of light in the night sky. A mysterious glowing object falls to earth. Residents of Town A and Town B see it fall.

The residents of Town A take a quick bearing on the object and measure its direction at 65°, measured (as bearings usually are) clockwise from true north. The quick-thinking residents of Town B do the same and determine that the object fell at a bearing of 40° when seen from their town. Town A is exactly 80 km northwest of Town B.

(a) Which town is closer to the place where the object fell?

(b) How far from the closest town did the object fall?

Question 17 *(Answer p190)*

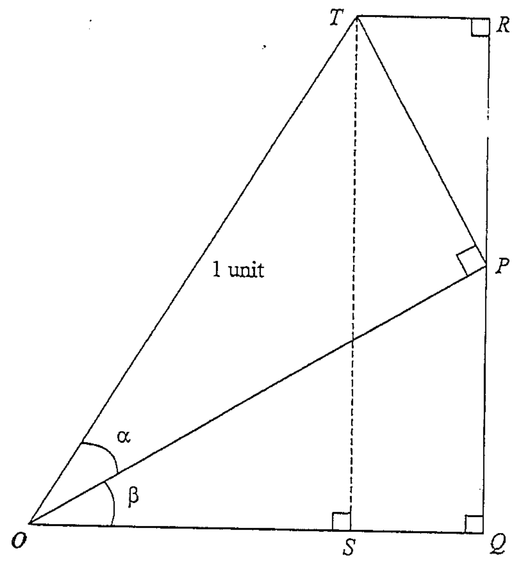
Solve for x. Answers **must** be accurate to at least **three** significant figures.

(a) $\tan(2x - 1) = 0.321$ where $0 \le x \le \frac{\pi}{2}$

(b) $\sin(x)\cos(x) = 0.321$ where $0 \le x \le \frac{\pi}{4}$

Question 19 *(Answer p191)*

The diagram below can be used to prove the compound angle formulae for sine and cosine in the acute angle case.



Use the diagram to prove ***EITHER*** the compound angle formula for sine ***OR*** the compound angle formula for cosine. Only elementary proofs that are based on this diagram will be accepted.

Question 22 *(Answer p194)*

Over a 24-hour period the depth $h(t)$ (in metres) of the water at a berth in a sheltered harbour varies with time t (in hours measured from midnight), according to the formula

$$h(t) = 3 + 2\sin(0.506t),\ 0 \le t \le 24.$$

Note that the sine function should be computed using radians and not degrees in this formula.

(a) What is the depth of the water at high tide?

(b) At what times (in hours after midnight) does high tide occur?

(c) What is the period of the function h?

Question 26 *(Answer p196)*

Prove the trigonometric identity $\tan(\theta) + \cot(\theta) = 2\,\mathrm{cosec}(2\theta)$.

SOLUTIONS

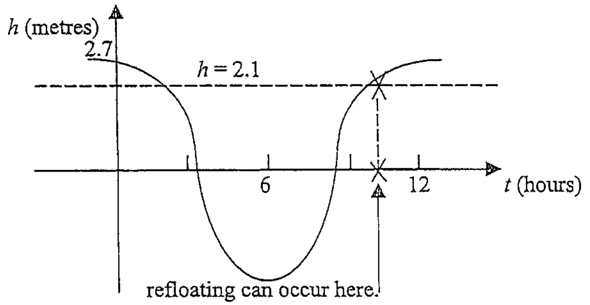
1 (c) Period = 8 (units) and Amplitude = 3 (units).

2 (b) Let x metres be the required distance. $\sin 50° = \frac{210}{200+x}$ $\therefore$ $(200+x)\sin 50° = 210$

so, $x = \frac{210 - 200\sin 50°}{\sin 50°} = 74.1\text{m}$ (3 sf)

(c) RHS $= \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\tan\theta}{\sec^2\theta} = 2\frac{\sin\theta}{\cos\theta} \times \frac{\cos^2\theta}{1} = 2\sin\theta\cos\theta = \sin 2\theta$ (= LHS)

(d) 1. and 2.



3. $2.7\cos\frac{\pi t}{6} = 2.1$ so $t = 1.3$ hours (or $12 - 1.3 = 10.7$ hours)

$\therefore$ time of refloating = 10.7 hours + 4.5 hours = 3.12 pm.

Question 3 (*Question p64)*

(a) 1. Amplitude = 2.4 or "a" and Period = 6 or $\frac{2\pi}{b}$.

2. a = 2.4 and b = $\frac{2\pi}{6}$ *or* $\frac{\pi}{3}$ or 1.05.

(b) 2 cos A sin B = sin (A+B) – sin(A-B).

2 cos 3Dsin D = sin(3D+D) – sin (3D-D) = sin 4D – sin2D.

(c) $(\tan\theta + \sec\theta)^2 = \left(\frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)^2 = \left(\frac{\sin\theta + 1}{\cos\theta}\right)^2 = \frac{(1+\sin\theta)^2}{\cos^2\theta}$

$= \frac{(1+\sin\theta)(1+\sin\theta)}{1-\sin^2\theta} = \frac{\cancel{(1+\sin\theta)}(1+\sin\theta)}{\cancel{(1+\sin\theta)}(1-\sin\theta)} = \frac{1+\sin\theta}{1-\sin\theta}$ (or in reverse).

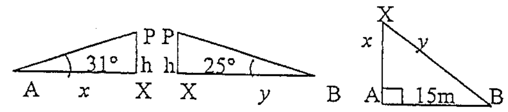
4 (c) $\sin 5x = \sin x$ so $\sin 5x - \sin x = 0 \Rightarrow 2\cos\frac{5x+x}{2}\sin\frac{5x-x}{2} = 0$

$\therefore$ $2\cos 3x \sin 2x = 0$, so either $\cos 3x = 0$ or $\sin 2x = 0$

$3x = \frac{\pi}{2}, \frac{3\pi}{2}$, or $\frac{5\pi}{2}$ so $x = \frac{\pi}{6}, \frac{\pi}{2}$, or $\frac{5\pi}{6}$.

$2x = 0, \pi$, or 2π so $x = 0, \frac{\pi}{2}$, or π. Therefore, $x = 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$.

(d)



From ΔPAX, $x = h\cot 31°$ or $\frac{h}{\tan 31°}$. From ΔPBX, $y = h\cot 25°$ or $\frac{h}{\tan 25°}$.

From ΔABX, using Pythagoras $x^2 + 15^2 = y^2$.

Hence $h^2\cot^2 31° + 15^2 = h^2\cot^2 25°$ or $h^2(\cot^2 25° - \cot^2 31°) = 225$.

$h^2 = \frac{225}{1.82908}$ so $h = 11.09\text{m}$

Total height of wall = 11.09 + 1.7 = 12.79m; Number of sheets = $\frac{12.79}{2.1} = 6.09$.

$\therefore$ They need 7 sheets (although 6 would probably do).

Question 5 (*Question p65)*

(a) 1. $\cos x = -0.6$ and $0 \le x \le 2\pi$ so $x = \cos^{-1}(-0.6)$

$x = \cos^{-1}(-0.6) = 2.214$ and $x = 2\pi - (\cos^{-1}(-0.6)) = 4.070$ so, $x = 2.214$, or 4.070. (4 sf).

2. $\sin 2x = 0.5$ $\quad 0^0 \le x \le 180^0$.

$2x = \sin^{-1}0.5$ so $x = \frac{\sin^{-1}0.5}{2}$ or $15°$; and $x = 90 - 15° = 75°$

so, $x = 15°, 75°$.

(b) 1. $y = A\sin B(x+C)$. $A = 2$, $B = 1$, and $C = \frac{\pi}{2}$

2. To prove $\sec^2 x + \operatorname{cosec}^2 x = \sec^2 x \operatorname{cosec}^2 x$

LHS $= \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x \sin^2 x}$

(but since $\sin^2 x + \cos^2 x \equiv 1$)

$= \frac{1}{\cos^2 x \sin^2 x} = \sec^2 x \operatorname{cosec}^2 x$ = RHS.

Question 7 *(Question p66)*

(a) 1. $x = 30°, 150°$

2. $x = 31.7°, 121.7°$

(b) 1. $B = 3.9 + 0.33 = 4.23$

2. $B = 3.9 - 0.33 = 3.57$

3. Period $= \dfrac{2\pi}{\left(2\pi/5.6\right)} = 5.6$ days.

(c) LHS: $\dfrac{\sin^2 x}{\cos^2 x} - \dfrac{\sin^2 x}{1} = \dfrac{\sin^2 x - \cos^2 x \sin^2 x}{\cos^2 x}$

$= \dfrac{\sin^2 x\left(1-\cos^2 x\right)}{\cos^2 x}$

$= \dfrac{\sin^2 x \sin^2 x}{\cos^2 x} = \tan^2 x \sin^2 x =$ RHS.

10 (b) 1. $x = -26.6°, 153.4°$ (1 dp)

2. $\sec(2x) = 2$

$\cos(2x) = 1/2$

$2x = 2n180° \pm 60°$

$x = 180n \pm 30°$

$n = 0 \Rightarrow x = \pm 30°$

$n = 1 \Rightarrow x = 150°, 210°$

$\therefore x = 30°, 150°$

(c) LHS $= \cot^2 x - \cos^2 x$

$= \dfrac{\cos^2 x}{\sin^2 x} - \cos^2 x$

$= \cos^2 x\left(\dfrac{1}{\sin^2 x} - 1\right)$

$= \cos^2 x\left(\dfrac{1-\sin^2 x}{\sin^2 x}\right)$

$= \cos^2 x\left(\dfrac{\cos^2 x}{\sin^2 x}\right)$

$= \cos^2 x \cdot \cot^2 x$

$=$ RHS

Question 11 *(Question p68)*

(a) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \dfrac{3/4 + 3/5}{1 - 3/4 \cdot 3/5}$

$= 2.45$ (3sf)

(b) $\cos^2 x - \sin x \cos x = 0$

$\cos x(\cos x - \sin x) = 0$

$\Rightarrow \cos x = 0$

$x = 2n\pi \pm \dfrac{\pi}{2}$

$n = 0 \Rightarrow x = \pm\dfrac{\pi}{2}$

$n = 1 \Rightarrow x = \dfrac{3\pi}{2}, \dfrac{5\pi}{2}$

or $\cos x - \sin x = 0$

$\cos x = \sin x$

$\therefore x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$ by considering the graphs of cos and sin.

$\therefore x = \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{5\pi}{4}, \dfrac{3\pi}{2}$.

(c) The function repeats every 12.5 hours

$\therefore B = \dfrac{2\pi}{12.5}$ (2π is the usual period of cos function)

$= \dfrac{4\pi}{25}$

let 10.00am be $t = 0$.

$\therefore A\cos(B \cdot 0) + C = 2$

$A + C = 2$ (1)

At low tide, $y = 0.6$

$\cos Bt =$ minimum

$= -1$

$\Rightarrow -A + C = 0.6$

$C = 0.6 + A$ (2)

Substitute (2) into (1): $A = 0.7$

Substitute into (2): $C = 1.3$

$\therefore y = 0.7\cos\left(\dfrac{4\pi}{25}t\right) + 1.3$

$y = 1 \Rightarrow 0.7\cos\left(\dfrac{4\pi}{25}t\right) + 1.3 = 1$

$\cos\left(\dfrac{4\pi}{25}t\right) = -0.429$ (3 sf)

$t = \dfrac{25}{4\pi}(2n\pi \pm 2.013)$

$= 25\left(\dfrac{n}{2} \pm 0.16\right)$

$n = 0 \Rightarrow t = 4hrs$ (time from 10am)

$n = 1 \Rightarrow t = 16.5, 8.5hrs$

$\therefore$ safe between 2pm and 6.30pm to walk across.

Question 13 *(Question p68)*

LHS $= \dfrac{\sin(3x)}{\sin(x)} - \dfrac{\cos(3x)}{\cos(x)}$

$= \dfrac{\sin(3x)\cos(x) - \cos(3x)\sin(x)}{\sin x \cos x}$

$= \dfrac{\sin(3x - x)}{\sin x \cos x}$ Using the identity sin(A – B) = sin A cos B – cos A sin B.

$= \dfrac{\sin(2x)}{\sin x \cos x}$

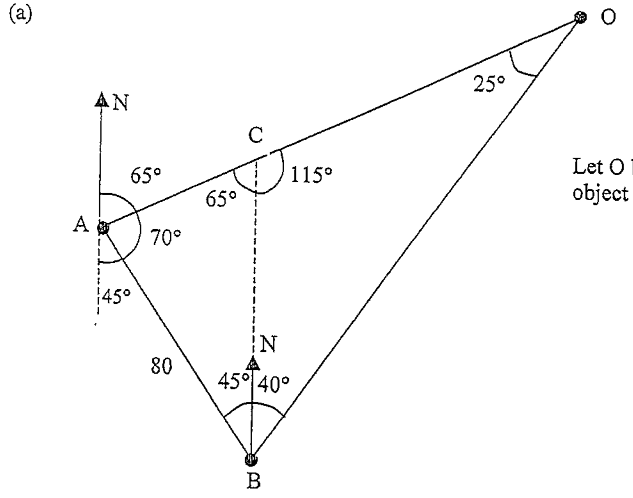
$= \dfrac{2\sin x \cos x}{\sin x \cos x}$ Using the identity sin2A = 2sinAcosA

$= 2$

$=$ RHS.

Question 14 *(Question p68)*

(a)



Let O be the point where the object falls.

Northwest is at a bearing of 135°, so $\angle OAB = 135° - 65° = 70°$. $\angle ACB = 65°$, because it and the bearing from Town A are alternate angles on parallel lines. $\angle ABC = 180° - 70° - 65° = 45°$.

$\angle OAB = 180° - 65° - 45° = 70°$

Since $\angle OAB = 70°$ is less than $\angle OBA = 85°$, the object is closer to town B.

(b) $\angle AOB = 180° - 70° - 85° = 25°$

Using the Sine Rule:

$$\frac{AB}{\sin(\angle AOB)} = \frac{OB}{\sin(\angle OAB)}$$

$$OB = \frac{80(\sin 70°)}{\sin 25°}$$

$$= 178 \text{ km (3 sf)}.$$

Question 17 *(Question p70)*

Note that in both parts (a) and (b), the small range of allowed x values means that there is only one solution.

(a) $\tan(2x-1) = 0.321\ ;\ 0 \le x \le \frac{\pi}{2}$

$$2x - 1 = \tan^{-1}(0.321)$$

$$x = \frac{\tan^{-1}(0.321)+1}{2}$$

$$= 0.655 \text{ (3 sf)}$$

(b) $\sin x \cos x = 0.321\ ;\ 0 \le x \le \frac{\pi}{4}$

$$\frac{1}{2}\sin(2x) = 0.321$$

$$x = \frac{\sin^{-1}(0.321\times 2)}{2}$$

$$= 0.349 \text{ (3 sf)}$$

Question 25 *(Question p73)*

(a) $\tan\alpha = \frac{a}{x}$ $\qquad \tan\beta = \frac{b}{x}$ $\qquad \theta = \beta - \alpha$

$$\tan\theta = \tan(\beta-\alpha)$$

$$= \frac{\tan\beta - \tan\alpha}{1+\tan\beta\tan\alpha}$$

$$= \frac{\frac{b}{x}-\frac{a}{x}}{1+\frac{b}{x}.\frac{a}{x}}$$

$$= \frac{x(b-a)}{x^2+ab}$$

Question 26 *(Question p73)*

$\tan\theta + \cot\theta = 2\operatorname{cosec}(2\theta)$

LHS $= \tan\theta + \cot\theta$

$$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$$

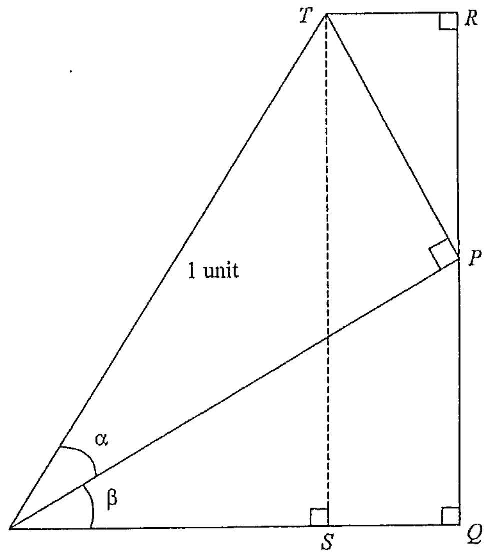
$$= \frac{1}{\frac{1}{2}\sin 2\theta - \frac{1}{2}\sin 0}$$

$$= \frac{1}{\frac{1}{2}\sin 2\theta}$$

$$= \frac{2}{\sin 2\theta}$$

$$= 2\operatorname{cosec}(2\theta) = \text{RHS}$$

Question 19 *(Question p71)*



Case 1: To prove that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

Firstly, TS = RQ, and $\sin(\alpha+\beta) = \text{TS}$.

So $\sin(\alpha+\beta) = \text{RQ}$

$= \text{RP} + \text{PQ}$ (Equation 1)

Secondly,

$$\sin\alpha\cos\beta + \cos\alpha\sin\beta = \frac{\text{TP}}{1}\times\frac{\text{OQ}}{\text{OP}}\times\frac{\text{OP}}{1}\times\frac{\text{PQ}}{\text{OP}}$$

$$= \frac{\text{TP}\times\text{OQ}}{\text{OP}} + \text{PQ} \quad \text{(Equation 2)}$$

Thirdly, triangles TRP and PQO are similar:

$\angle$PQO and $\angle$TRP are both 90°.

$\angle\text{PQO} + \angle\text{QOP} + \angle\text{OPQ} = 180°$

$90° + \angle\text{TPR} + \angle\text{OPQ} = 180°$

$\therefore \angle\text{QOP} = \angle\text{TPR}$

And if $\angle\text{PQO} = \angle\text{TRP}$ and $\angle\text{QOP} = \angle\text{TPR}$, then $\angle\text{OPQ} = \angle\text{PTR}$.

Now, $\beta = \angle\text{QOP} = \angle\text{TPR}$, so it must follow that

$\cos\beta = \cos(\angle\text{QOP}) = \cos(\angle\text{TPR})$,

$\cos\beta = \frac{\text{OQ}}{\text{OP}} = \frac{\text{RP}}{\text{TP}}$

If $\frac{\text{OQ}}{\text{OP}} = \frac{\text{RP}}{\text{TP}}$, then $\frac{\text{TP}\times\text{OQ}}{\text{OP}} = \text{RP} = \sin\alpha\cos\beta$.

So $\sin\alpha\cos\beta + \cos\alpha\sin\beta = \text{RP} + \text{PQ}$.

Equation 1 and Equation 2 are the same, so $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ as required.

Case 2: To prove that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$\cos(\alpha+\beta) = \frac{\text{OS}}{\text{OT}}$$

$$= \text{OS}$$

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{OP}{OT}\times\frac{OQ}{OP} - \frac{PT}{OT}\times\frac{PQ}{OP}$$

$$= \text{OQ} - \frac{\text{PT}\times\text{PQ}}{\text{OP}}$$

Since the triangles TRP and POQ are similar,

$$\frac{\text{TR}}{\text{PT}} = \frac{\text{PQ}}{\text{OP}}$$

$$\text{TR} = \frac{\text{PT}\times\text{PQ}}{\text{OP}}$$

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta = \text{OQ} - \text{TR}$$

$$= \text{OS}$$

$= \cos(\alpha+\beta)$, as required.

Question 22 *(Question p72)*

(a) The maximum value of $\sin x$ is 1, so the depth at high tide will be $3 + 2\times 1 = 5$ m.

(b) $\sin x = 1$ when $x = n\pi + (-1)^n\frac{\pi}{2}$

$0.506t = n\pi + (-1)^n\frac{\pi}{2}$

$t = \frac{n\pi}{0.506} + (-1)^n\frac{\pi}{2\times 0.506}$

$\approx 3.104, 15.52$ hours (4 sf)

(c) The period of the function h is equal to the time between high tides.

period $= 15.52 - 3.104$

$= 12.4$ hours (3 sf)

The period can also be calculated from the formula, since the period of $\sin\alpha x$ is $\frac{2\pi}{\alpha}$.

period $= \frac{2\pi}{0.506}$

$= 12.4$ hours (3 sf)